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# Survey of roughness by stochastic oscillations

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## Abstract

In this paper, connection between surface roughness and directed polymers in random medium are studied, when the surface is considered as a directed line undergoing stochastic oscillations. This is performed by studying the influence of a stochastic elastic forcing term  $-\kappa y + \eta(s)$ , on a particle moving along a rough surface. Two models are proposed and analysed in this way: the random-walk process (RW) in its discrete and continuous form, and a Markovian process via the Ornstein-Uhlenbeck (O-U) process. It is shown that the continuous RW leads to an oscillator equation, via an effective action obeying a KPZ equation which is solved analytically. The O-U process allows to obtain information on the profile of surface for a long size substrate. The analogy with the roughness is achieved by introducing a quantity suited to directed line formalism: the height velocity variation  $\partial h / \partial s$ .

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## I. INTRODUCTION

Characterization of surface roughness or interface growth is of first importance in many applications: microelectronics, preparation of catalysts, magnetic materials (see *e.g.* [1]-[4]). And numerous models based on numerical as well as analytical approaches have been proposed over past decades to treat interface properties. These works concern in particular the KPZ equation (scale laws, roughness, growth) [5]-[15], and models treating connections between surface growth and directed polymers in random medium [16]-[24]. The present paper focus on the second class of these models, considering the roughness of a surface as the result of specific stochastic processes, and based on the directed line formalism. This formalism, much used in polymer physics [25], allows to analyse the motion of a system by the way of transversal and curvilinear coordinates [26]-[28] (respectively noted  $y$  and  $s$  in this paper). And we propose models showing the analogies between the profile of a rough surface and the study of the transversal position equation. Note that, in this formalism,  $s$  plays the role of the time. For example  $dy/ds$  designs the transversal velocity. Moreover, this formalism is also linked to the statistical physics: the partition function of the system and its links with the KPZ or the Burgers equation, which follow from this formalism, represent useful tools [29]-[31]. Concerning the roughness analysis, different works have treated this problem [29], [32]-[37], but the roughness is nevertheless poor studied from a purely stochastic point of view. Among recent works treating stochastic surface properties, we can cite [38] and [39] which both have a Markovian approach, using a Langevin equation with a stochastic noise.

In the models presented here, we assumed that the profile of surface is described by the trajectory of a particle undergoing a directing stochastic strength in addition to diffusion. We will use for this purpose the elastic forcing term  $f(y, s) = -\kappa y + \eta(s)$  inducing stochastic oscillations, or equivalently, a potential  $\Phi$ , such as  $f = -\partial\Phi/\partial y$ ;  $\eta$  having for example the properties of a Langevin force. For deposition processes, we suppose that the surface start at the height  $h_0$ , and we are interested in the height velocity variation (hvv),  $\partial h/\partial s$ , which gives local information on the structure of the surface to analyse: for a fast variation of the height, we will have a high value of the hvv, (Fig. (1)). Hence, this quantity is the reflect of local irregularities. But, the hvv do not have to be confused with the quantity  $\partial h/\partial x$ , where  $x$  stands for the horizontal coordinate: locally, the slope  $\partial h/\partial x$  may takes an infinite value (*i.e* parallel to the  $h$  axis), whereas  $(h(s_1) - h(s_2))/(s_1 - s_2)$  is weak.

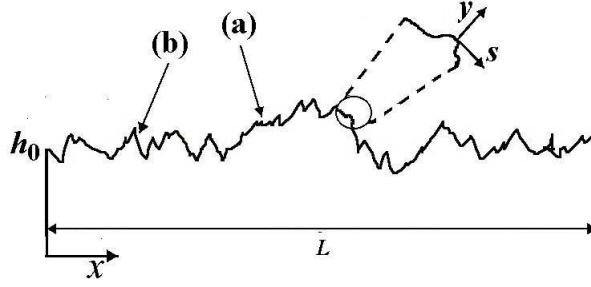


FIG. 1: Profile of surface: result of the conflict between the random medium and a directing strength  $-\partial\Phi/\partial y$ . The local slope (a) (resp. (b)) stands for a weak (resp. high) value of the hvv.

The outline of the paper will be as follow: we first derive properties related to profile of surface from pure and forced Random Walk (RW). In what follows, we will employed the term forced RW to design a RW undergoing a forcing term in addition to the diffusion. Then, we justify the use of an oscillator model comparing the directed line to a string vibrating. This will allow to use the restricted partition function to determine the transversal equation of motion, in presence of a quadratic stochastic potential. In the last part, we shall concentrate on the O-U equation of motion for the transversal position, and particularly on the asymptotic solutions for  $s \rightarrow \infty$ .

## II. TOPOLOGY OF SURFACE VIEW AS A RANDOM WALK PROCESS

The models presented here concern the use of the directed line formalism associated to a RW process. In its discrete form, we based our approach on the paper of Vilgis [25], treating directed polymers in random media. Our derivation is somewhat different of Vilgis, but leads to the well-known partition function of polymer physics as showed later on.

### A. Discrete random walk process

We start from a set of vectors  $\{\mathbf{b}_i \equiv b_i \mathbf{e}_s; i = 1, \dots, N\}$ , which are successively connected, and recovering the whole of the surface (Fig. (2)). We suppose they are statistically independent.

The probability to find such a set is given by:

$$P(\{\mathbf{b}_i; i = 1, \dots, N\}) = \prod_{i=1}^N p(\mathbf{b}_i), \quad (1)$$

where the probability  $p(\mathbf{b}_i)$  corresponds to a step of the random walk of  $\mathbf{b}_i$ , with the mean value

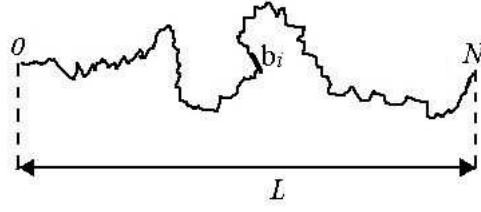


FIG. 2: Random Walk:  $N$  steps of length  $b$  are required in order to cover the surface on a linear size substrate  $L$ .

properties:

$$\langle \mathbf{b}_i \rangle = 0 \quad \langle \mathbf{b}_i \mathbf{b}_j \rangle = b^2 \delta_{ij}. \quad (2)$$

$\delta_{ij}$  is the Kronecker symbol and  $b$  is a characteristic length of the shift of position of the surface from one to the other. For the step  $\mathbf{b}_i$ , a normalized gaussian probability law is taken:

$$p(\mathbf{b}_i) = (1/2\pi b^2)^{1/2} \exp(-\mathbf{b}_i^2/2b^2), \quad (3)$$

and the associated distribution of the surface length reads

$$P(\mathbf{R}) = \int d\mathbf{b}_1 \dots \int d\mathbf{b}_N \delta \left( \mathbf{R} - \sum_{i=1}^N \mathbf{b}_i \right) \prod_{i=1}^N p(\mathbf{b}_i); \quad (4)$$

the vector  $\mathbf{R} = \sum_{i=1}^N \mathbf{b}_i$  characterizing the line. Then, using the Fourier transform representation of the Dirac delta, a Gaussian law for the length distribution arises quite naturally

$$P(\mathbf{R}) = (1/2\pi b^2 N)^{1/2} \exp(-\mathbf{R}^2/2b^2 N). \quad (5)$$

And from this expression, the quadratic mean “length” of the surface can be evaluated:

$$\langle \mathbf{R}^2 \rangle = \int \mathbf{R}^2 P(\mathbf{R}^2) d\mathbf{R} = b^2 N. \quad (6)$$

Note that the quantity  $\langle \mathbf{R}^2 \rangle$  preserves information on the length of the trajectory, contrary to the geometrical quantity  $\mathbf{R}$ . However, this result is a limiting case of a pure RW trajectory since it implies that the entanglements, and so the length of the surface, could escape to infinity. Thereby, for this case we label  $N$  such as

$$\langle \mathbf{R}^2 \rangle_{\text{rw}} = b^2 N_{\text{rw}}, \quad (7)$$

with the meaning that  $N_{\text{rw}}$  steps are necessary in average for a pure RW to describe the profile of surface. The other limiting case, *i.e.* the straight line (sl), is given by

$$\langle \mathbf{R}^2 \rangle_{\text{sl}} = b^2 N_{\text{sl}}^2 \simeq L^2, \quad (8)$$

in such a way that the general case (g), representing a forced RW, and included between the cases (7) and (8), may be written

$$\langle \mathbf{R}^2 \rangle_{\text{g}} = b^2 N_{\text{g}}^{2\gamma}; \quad N_{\text{rw}} \leq N_{\text{g}} \leq N_{\text{sl}}, \quad (9)$$

where  $\gamma$  is a critical exponent representative of a strength acting on the RW. In addition, we have the conditions

$$b^2 N_{\text{g}}^{2\gamma} \rightarrow b^2 N_{\text{sl}}^2 \quad \text{for} \quad N_{\text{g}} \rightarrow N_{\text{sl}}; \quad (10)$$

which implies  $\gamma = 1$ , and

$$b^2 N_{\text{g}}^{2\gamma} \rightarrow b^2 N_{\text{rw}} \quad \text{for} \quad N_{\text{g}} \rightarrow N_{\text{rw}}; \quad (11)$$

implying  $\gamma = 1/2$ . Thus,  $1/2 < \gamma < 1$ , available for a line moving in the plane  $(h, x)$ . This result may also be expressed in term of the length  $l \simeq \sqrt{\langle \mathbf{R}^2 \rangle_{\text{g}}}$  of the surface:

$$l = b N_{\text{g}}^{\gamma}. \quad (12)$$

In the case of a two-dimensional RW,  $\gamma$  may takes the form  $\gamma = 1/d_{\text{f}}$ , where  $d_{\text{f}}$  designs the fractal dimension of the directed line. As a matter of fact,  $d_{\text{f}}$  describes the degree of irregularity of the surface, and for  $\gamma = 1$  and  $\gamma = 1/2$ ,  $d_{\text{f}}$  becomes identical to the topological dimension of the system. Clearly, the exponent  $\gamma$  is linked to the number of steps  $N_{\text{g}}$ . A relation between the two quantities may be obtained from the probability of the general case,  $P_{\text{g}}(\mathbf{R})$ , when the line undergoes both effects of diffusion and of an external force. Anticipating the continuum approach described afterwards, we write this probability

$$P_{\text{g}}(\mathbf{R}) = \mathcal{N} \exp \left( -\mathbf{R}^2 / 2b^2 N_{\text{g}} - \beta \Phi(\mathbf{R}) \right), \quad (13)$$

where  $\Phi(\mathbf{R})$  stands for an external potential. For example, in the case of a harmonic potential,  $\Phi = (\kappa/2)\mathbf{R}^2$ , the quadratic mean length is

$$\langle \mathbf{R}^2 \rangle_{\text{g}} = \int \mathbf{R}^2 P_{\text{g}}(\mathbf{R}^2) d\mathbf{R} = \frac{b^2 N_{\text{g}}}{1 + b^2 \beta \kappa N_{\text{g}}} \quad (14)$$

which leads, with Eq. (9), to the expression

$$\gamma = \frac{\ln(N_g/(1 + b^2\beta\kappa N_g))}{2 \ln(N_g)}. \quad (15)$$

One can then obtain information on the topology of a surface, in the sense that the ratio  $l/L$  is the reflection of its irregularities and roughness. The fractal dimension may be evaluated experimentally, for example, by the box-counting method with boxes of length  $b$  (in the same way as for electric discharges [40]). In simulations, the varying parameter would be the number of steps of the RW.

## B. Equations of evolution

If we express now the step as  $\mathbf{b}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$ , and writing shortly  $P = P(\{\mathbf{b}_i; i = 1, \dots, N\})$ , the probability (Eq. (1)) may be rewritten

$$P = \left(\frac{1}{2\pi b^2}\right)^{N/2} \exp\left[-\frac{1}{2b^2} \sum_{i=1}^N (\mathbf{R}_i - \mathbf{R}_{i-1})^2\right]. \quad (16)$$

One can then associated to the exponential argument a symbolic Hamiltonian

$$\beta H_0 = \frac{1}{2b^2} \sum_{i=1}^N (\mathbf{R}_i - \mathbf{R}_{i-1})^2. \quad (17)$$

This is the energy of a mass-spring chain of a one dimensional lattice. This suggest to take the continuous limit of this discrete chain:  $\mathbf{R}_i - \mathbf{R}_{i-1} \rightarrow \partial\mathbf{R}/\partial s$ , and  $\sum_{i=1}^N \rightarrow \int_0^N$ , where  $s$  is the dimensionless curvilinear coordinate. In this limit, the probability of the pure RW reads

$$P \rightarrow \mathcal{N} \exp\left[-\frac{1}{2b^2} \int_0^N \left(\frac{\partial\mathbf{R}}{\partial s}\right)^2 ds\right], \quad (18)$$

where  $\mathcal{N}$  is a normalization constant. From this result, one can construct the partition function as a sum over all the possible paths starting from  $s = 0$  and reaching the linear size substrate at  $s = N$ :

$$Z = \mathcal{N} \sum_{\text{All paths } \mathbf{R}(s)} \exp\left[-\frac{1}{2b^2} \int_0^N \left(\frac{\partial\mathbf{R}}{\partial s}\right)^2 ds\right]. \quad (19)$$

The argument of the exponential is equivalent to the transversal kinetic energy, or more precisely to the effective action of the system since we integrate over the “time”  $s$ . Furthermore, this path integral obeys the classical diffusion equation which arises naturally from a pure RW process:

$$\frac{\partial}{\partial s} Z(\mathbf{R}, s) = \frac{b^2}{2} \frac{\partial^2}{\partial R^2} Z(\mathbf{R}, s), \quad (20)$$

with the boundary condition  $Z(\mathbf{R}, 0) = \delta(\mathbf{R})$ . For a forced RW, we introduce a potential term  $\Phi(\mathbf{R}(s))$  standing for an external (virtual) strength field, in such a way that the total effective action takes the form

$$\beta\widehat{\mathcal{S}} = \frac{1}{2b^2} \int_0^N \left( \frac{\partial \mathbf{R}(s)}{\partial s} \right)^2 ds + \int_0^N \Phi(\mathbf{R}(s)) ds. \quad (21)$$

We note  $\widehat{\mathcal{S}}$  to emphasize the fact that this quantity represents a pseudo action (the integration is made over a Hamiltonian and not over a Lagrangian). This yields to the total partition function of the system, as the Feynman-Kac path integral:

$$Z(\mathbf{R}, s) = \mathcal{N} \int \exp \left[ -\beta\widehat{\mathcal{S}} \right] \mathcal{D}\mathbf{R}(s). \quad (22)$$

It is then convenient to introduce the transversal coordinate  $\mathbf{y} \equiv \mathbf{R}/b$  and so to write the action in the form [26]

$$\widehat{\mathcal{S}} = \int_0^s ds' \left( \frac{1}{2} \left( \frac{d\mathbf{y}}{ds'} \right)^2 + \Phi(\mathbf{y}, s') \right) = \int_0^s ds' H(\mathbf{y}, s'). \quad (23)$$

$H$  standing for the Hamiltonian of the system. By this way, we obtain the well-know expression of the partition function of directed polymers

$$Z(\mathbf{y}, s) = \mathcal{N} \int_{\mathbf{y}(0)}^{\mathbf{y}(s)} \mathcal{D}\mathbf{y} \exp \left[ -\beta \int_0^s ds' \left( \frac{1}{2} \left( \frac{d\mathbf{y}}{ds'} \right)^2 + \Phi(\mathbf{y}, s') \right) \right]; \quad (24)$$

expression directly issued from a RW process with a directed line formalism. For a one dimensional system,  $Z(y, s)$  obeys the following PDE [41]:

$$\frac{\partial}{\partial s} Z(y, s) = \nu \frac{\partial^2 Z(y, s)}{\partial y^2} - \beta \Phi(y, s) Z(y, s), \quad (25)$$

where  $\nu = 1/2\beta$  is a viscosity coefficient.

Let us underline that there exists several versions of the Hamiltonian depending on system to study [16]-[18], [20]-[24].

In view of these results, several analogies may be drawn between directed lines and vibrating strings. In its discrete form, the directed line is composed of a succession of rigid chains and the whole evolving with a resulting motion depending on the form of the noisy potential. For the continuous form, the term of inertia in polymer physics is due to the thermal agitation of the medium, and the potential to an external field (*e.g.* electric field). In our case, the external potential is necessary because of the strength which have to be imposed to the line, in order to obtain realistic profiles of surface. So, the noisy potential of our model will be composed of a harmonic part and



of a stochastic part which shall make the oscillations stochastic:  $\Phi(y, s) = \frac{1}{2}\kappa y^2 + \eta(y, s)$ , where  $\kappa$  is the elastic constant of the line. The next step is to determine the equation of motion with the frequency of vibration of the line. In order to do this, let us proceed as follow:

In one hand, starting from Eq. (25), it is well-known that we may obtain successively a KPZ and a Burgers equation (approach (A)):

$$\frac{\partial \hat{S}}{\partial s} = \nu \frac{\partial^2 \hat{S}}{\partial y^2} + \frac{1}{2} \left( \frac{\partial \hat{S}}{\partial y} \right)^2 - \Phi; \quad (26)$$

and

$$\frac{\partial u}{\partial s} + u \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\partial \Phi}{\partial y}. \quad (27)$$

In this case the transformations are

$$u = -\frac{\partial \hat{S}}{\partial y} \quad \text{and} \quad Z = \exp \left( \hat{S}/2\nu \right). \quad (28)$$

Usually one introduce the free energy. But the free energy tends to the total energy of the system  $\hat{S}$  when the entropy tends to zero.

In the other hand, starting from a general Burgers equation for a velocity field  $u$ , we have (approach (B))

$$\frac{\partial u}{\partial s} + u \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\partial V}{\partial y}, \quad (29)$$

and

$$\frac{\partial F}{\partial s} = \nu \frac{\partial^2 F}{\partial y^2} + \frac{1}{2} \left( \frac{\partial F}{\partial y} \right)^2 + V, \quad (30)$$

and finally, with  $\hat{Z} = \exp (F/2\nu)$ ,

$$\frac{\partial}{\partial s} \hat{Z}(\mathbf{x}, s) = \nu \nabla^2 \hat{Z}(\mathbf{x}, s) + \frac{V}{2\nu}. \quad (31)$$

Whether  $u$  represents the transversal velocity, we have in this case

$$u = \frac{dy}{ds} = -\frac{\partial F}{\partial y}. \quad (32)$$

From these two approaches, a question arises: “May the quantity  $u$  of Eq. (27) be interpreted as the transversal velocity of Eq. (29) ?” In other words, is there identity between  $\hat{S}$  and  $F$  ?

To remove this ambiguity, let us start from the Burgers equation (29),  $u$  given by (32), and let us use the relation

$$dF(y, s) = \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial y} dy; \quad (33)$$

which yields to

$$dF = \nu \frac{\partial^2 F}{\partial y^2} ds + \frac{1}{2} \left( \frac{\partial F}{\partial y} \right)^2 ds + V ds + \frac{\partial F}{\partial y} dy; \quad (34)$$

and consequently to the expression

$$F = \nu \int \frac{\partial^2 F}{\partial y^2} ds + \frac{1}{2} \int \left( \frac{\partial F}{\partial y} \right)^2 ds + \int V ds + \int \frac{\partial F}{\partial y} dy. \quad (35)$$

The first and last integrals may be rewritten

$$\int \frac{\partial^2 F}{\partial y^2} ds = \frac{\partial}{\partial y} \int \frac{\partial F}{\partial y} ds = -\frac{\partial}{\partial y} \int \frac{dy}{ds} ds = -1. \quad (36)$$

and

$$\int \frac{\partial F}{\partial y} dy = - \int \frac{dy}{ds} dy = - \int \left( \frac{dy}{ds} \right)^2 ds, \quad (37)$$

In such a way that the “free energy” will read

$$F(y, s) = -\nu - \int_0^s \left[ \frac{1}{2} \left( \frac{dy}{ds'} \right)^2 - V(y, s') \right] ds'. \quad (38)$$

Finally, the Hopf-Cole transformation  $\widehat{Z} = \exp(F/2\nu)$ , with  $1/2\nu = \beta$ , yields explicitly to

$$\widehat{Z} = e^{-1/2} \times \exp \left\{ -\beta \int_0^s ds' \left( \frac{1}{2} \left( \frac{dy}{ds'} \right)^2 - V(y, s') \right) \right\}. \quad (39)$$

The heat equation (25) being invariant under the shift  $Z \rightarrow \widehat{Z} = c \times Z$ ,  $c \in \mathbb{R}$ , we will have equivalence between the two approaches if  $\Phi = -V$ . And, therefore, one obtain the classical Hamilton equation of motion:

$$\frac{dy}{ds} = -\frac{\partial F}{\partial y} = -\frac{\partial \widehat{S}}{\partial y} \quad \Longleftrightarrow \quad \frac{d^2 y}{ds^2} = -\frac{\partial H}{\partial y} \quad (40)$$

The determination of an equation of motion for  $y$  is henceforth equivalent to solve the KPZ equation (26). Moreover, we have a relation between the height of the growing surface and the transversal position:  $|h(s) - h_0| \equiv |y(s) - y(0)|$ , and so, by extension,

$$\frac{dy}{ds} \simeq \frac{\partial h}{\partial s}. \quad (41)$$

For example  $\partial h / \partial s = 0 \Leftrightarrow dy / ds = 0$ . Thereby, the action may as well be written [42, 43]

$$\widehat{S} = \int_0^s \left( \frac{1}{2} \left( \frac{\partial h}{\partial s'} \right)^2 + \Phi(h, s') \right) ds', \quad (42)$$

We shall now concentrate on the solution of the KPZ equation (26).

Analytic solutions of noisy KPZ equations have already been studied in the literature in specific cases [44]-[46]. But the non-linearity of this equation added to the presence of a stochastic noise make the resolution difficult. Nevertheless, it is known that this equation may be transformed either into a Burgers equation or a Schrödinger equation with an imaginary time. And, among works which treat exact solutions of these kind of equations, we can cite [47]-[51]. This is our concern here to treat the case of a line undergoing the stochastic elastic forcing term  $-\kappa y + \eta(s)$ , where the function  $\eta$  contains all the noise that affects the oscillations (*e.g.* a Langevin force). This amounts to saying that we solve the KPZ equation (26) with a potential  $\Phi$  given by  $\Phi(y, s) = \frac{\kappa}{2}y^2 - y\eta(s)$ . Moreover, we assume a constant initial condition for the transversal velocity:  $u(y, 0) = u_0$ .

Following methods presented in Ref. [51], it is shown in appendix that the solution is of the form:

$$\widehat{S}(y, s) = \frac{\xi(s)}{2}y^2 - y\psi(s) + \Theta(s), \quad (43)$$

where  $\xi(s)$ ,  $\psi(s)$  and  $\Theta(s)$  are functions of  $s$  and of  $\eta(s)$  (see the appendix). But, we have the relation

$$\frac{dy}{ds} = -\frac{\partial \widehat{S}}{\partial y} = -y\xi(s) + \psi(s). \quad (44)$$

Consequently, we obtain an equation standing for the equation of a forced oscillator with a term of friction:

$$\frac{d^2y}{ds^2} + \xi(s)\frac{dy}{ds} + \omega^2(s)y = F(s), \quad (45)$$

where the pulsation  $\omega^2$  and the forcing term  $F$  are respectively given by  $d\xi/ds$  and  $d\psi/ds$ . We can notice from the expressions of  $\xi$  and  $\psi$  that, while the frequency  $f \propto \sqrt{d\xi/ds}$  depends only on  $s$ , the term of force depends also on the stochastic noise  $\eta$ . Thus, we expect profiles of the form drawn in Fig. (3).

Once  $y$  determined, we can deduce the hvv from Eq. (44):

$$\frac{\partial h}{\partial s} \simeq -\xi(s) \exp\left(-\int_0^s \xi(t)dt\right) \left\{1 + \int_0^s \psi(t) \exp\left(\int_0^t \xi(t')dt'\right) dt\right\} + \psi(s). \quad (46)$$

Then, the degree of irregularity of a surface, and so its global roughness, may be estimated by adding up all the local slopes  $|\partial h/\partial s|$ , in such a way that the interesting quantity to determine is  $\langle |\partial h/\partial s| \rangle$ , where the mean is taken over all the length of the surface.

Note that this problem is different of the one of a stochastic oscillator, which is a specific problem [52, 53]

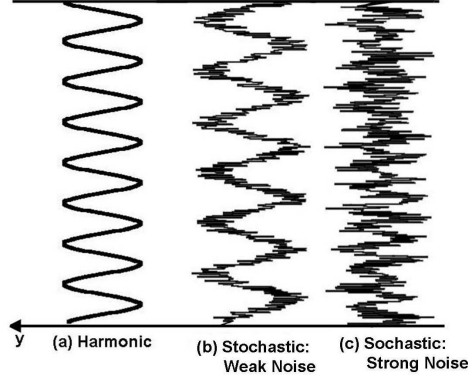


FIG. 3: Variations of the transversal position  $y$  for a forcing term inducing oscillations depending on the intensity of the noise. Each mode (a), (b), and (c) leads to very different roughness.

### III. LONG SIZE SUBSTRATE

We treat now the case of a particle of mass unity, moving along a rough surface when we study its profile at a fixed time.

Let us consider the propagation of a Brownian particle through virtual scatterers. Its trajectory describing the surface to analyse. We assume that the particle obeys a general Langevin equation where an external force  $-\partial\Phi/\partial y = -\tilde{\kappa}y$  is considered:

$$\frac{d^2y}{ds^2} = -\xi \frac{dy}{ds} - \tilde{\kappa}y + \Gamma(s). \quad (47)$$

This means that the particle undergoes effects of the medium (term of friction) in addition with an oscillator term. Then, the Langevin force  $\Gamma(s)$  may be expressed in term of a Gaussian with noise  $\eta(s)$ , such as

$$\frac{d^2y}{ds^2} = -\xi \frac{dy}{ds} - \tilde{\kappa}y + \sqrt{2\mathcal{D}}\eta(s); \quad (48)$$

where  $\mathcal{D} = \xi^2\tilde{\nu}$ . By nature, the solution of this equation follows a Markovian process. When the inertial term is negligible, one obtain an Ornstein-Uhlenbeck process for the transversal position  $y$ :

$$\frac{dy}{ds} = -\frac{\tilde{\kappa}}{\xi}y + \sqrt{2\tilde{\nu}}\eta(s). \quad (49)$$

This well-known process [54, 55] is associated to the following Fokker-planck equation for the transition probability  $\tilde{P}(y, s|y', s')$ :

$$\frac{\partial \tilde{P}}{\partial s} = \frac{\tilde{\kappa}}{\xi} \frac{\partial y \tilde{P}}{\partial y} + \tilde{\nu} \frac{\partial^2 \tilde{P}}{\partial y^2}. \quad (50)$$

With the standard initial condition  $\tilde{P}(y, s|y_0, s_0) = \delta(y - y')$ , the solution writes [56]

$$P(y, y', s) = \sqrt{\frac{\tilde{\kappa}}{2\pi\xi\tilde{\nu}(1 - e^{-(2\tilde{\kappa}/\xi)s})}} \exp \left[ -\frac{\tilde{\kappa}(y - e^{-(2\tilde{\kappa}/\xi)s}y')^2}{2\xi\tilde{\nu}(1 - e^{-(2\tilde{\kappa}/\xi)s})} \right]. \quad (51)$$

This probability is asymptotically “stationary”:

$$\lim_{s \rightarrow \infty} \tilde{P}(y, y', s) = \sqrt{\frac{\tilde{\kappa}}{2\pi\xi\tilde{\nu}}} \exp \left[ -\frac{\tilde{\kappa}y^2}{2\xi\tilde{\nu}} \right]. \quad (52)$$

In practice, the condition  $s \gg \xi/2\tilde{\kappa}$  is sufficient to obtain this solution. This implies that, under this condition, the fluctuations of the transversal position  $y$  becomes independent of the curvilinear abscissa. Moreover, the quantity  $\Lambda = \sqrt{\xi\tilde{\nu}/\tilde{\kappa}}$ , appearing in Eq. (52) with the dimension of a length, takes the sense of the mean range for the transversal fluctuations with respect to a given position:

$$\int_{-\infty}^{+\infty} y^2 \tilde{P}(y) dy = \Lambda^2. \quad (53)$$

And the Einstein relation  $\tilde{\nu} = 1/\xi\beta$  gives us the expression

$$\Lambda \propto \tilde{\kappa}^{-1/2} \quad (54)$$

Thereby, the higher the elastic constant, the lower the range of fluctuations. The connection between the O-U process and the roughness is then highlighted in Fig. (4), where we can notice that, for  $s \gg 1$ , the maximal fluctuation values of the height  $h(y, s)$  are of the order of  $\Lambda$ . As a consequence, the hvv will be such as

$$\left| \frac{\partial h}{\partial s} \right| \leq \Lambda \quad \forall s \gg \xi/2\tilde{\kappa}. \quad (55)$$

Consequently, in this limit,

$$\left\langle \left| \frac{\partial h}{\partial s} \right| \right\rangle \leq \Lambda; \quad (56)$$

available for models concerning processes of deposition executed from the left to the right of the figure, and on a “large” length of substrate.

Then, an interesting point lies in a connexion between the O-U process and KPZ equations. Indeed, the transformation

$$\tilde{P}(y, s) = \tilde{Z}(y, s)e^{-\tilde{\kappa}y^2/4\tilde{\nu}}, \quad (57)$$

applied to Eq. (50), leads to the heat equation

$$\frac{\partial \tilde{Z}}{\partial s} = \tilde{\nu} \frac{\partial^2 \tilde{Z}}{\partial y^2} + \left( \frac{\tilde{\kappa}}{2} - \frac{\tilde{\kappa}^2 y^2}{4\tilde{\nu}} \right) \tilde{Z}, \quad (58)$$

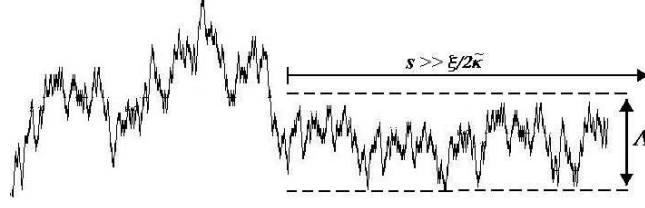


FIG. 4: Asymptotic profile of surface for an O-U process.  $\Lambda$  has to be interpreted as the maximal range of fluctuations.

and so to the KPZ equation

$$\frac{\partial \tilde{S}}{\partial s} = \tilde{\nu} \frac{\partial^2 \tilde{S}}{\partial y^2} + \frac{1}{2} \left( \frac{\partial \tilde{S}}{\partial y} \right)^2 + \left( \frac{\tilde{\kappa}}{2} - \frac{\tilde{\kappa}^2 y^2}{4\tilde{\nu}} \right). \quad (59)$$

And once more, the introduction of a quantity  $u$  such as  $u = -\partial \tilde{S} / \partial y$ , yields to a Burgers equation:

$$\frac{\partial u}{\partial s} + u \frac{\partial u}{\partial y} = \tilde{\nu} \frac{\partial^2 u}{\partial y^2} - \frac{\partial \tilde{\Phi}}{\partial y}. \quad (60)$$

But the same interrogation as the one of the RW process may be done, namely, may  $u$  be considered as the velocity  $dy/ds$  of Eq. (49) ? As a matter of fact, these relations have been obtained with the assumption  $d^2y/ds^2 \simeq 0$ . So, let us assume that  $u = dy/ds$ . The consequence is that the first term of Eq. (60) vanishes, and so we obtain

$$\begin{aligned} \frac{1}{2} \frac{\partial \tilde{u}^2}{\partial y} &= \tilde{\nu} \frac{\partial^2 \tilde{u}}{\partial y^2} - \frac{\partial \tilde{\Phi}}{\partial y} \\ \Leftrightarrow \frac{\partial}{\partial y} \left\{ \frac{1}{2} \left( \frac{dy}{ds} \right)^2 + \tilde{\Phi} \right\} &= \frac{\partial H}{\partial y} = \tilde{\nu} \frac{\partial^2}{\partial y^2} \left( \frac{dy}{ds} \right). \end{aligned} \quad (61)$$

Furthermore, we have that  $-\partial H / \partial y = d^2y/ds^2 \simeq 0$ . So the last part of Eq. (61), leads to a solution of the form

$$\frac{dy}{ds} = c_1(s)y + c_2(s). \quad (62)$$

Thus,  $u = dy/ds \Leftrightarrow u$  obeys to Eq. (62). Then, we refound Eq. (49) by putting  $c_1 = -\tilde{\kappa}/\xi$  and  $c_2(s) = \sqrt{2\tilde{\nu}\eta}(s)$ .

Once again, a stochastic process (here Markovian) may be connected to the coupled KPZ/Burgers equations, and by extension to the Hamiltonian of the system. In addition, a partition function like the one of the RW may be obtained with appropriated  $\tilde{\nu}$  and  $\tilde{\kappa}$ . Moreover, this shown the relevant choice of a harmonic potential  $\sim y^2$  to treat these kind of fluctuations, since a Markovian process and a RW process yields both to the same class of equations.

#### IV. CONCLUSION

Two short stochastic models of directed line have been presented and applied to the description of rough surfaces. The background of our approach was that roughness of a one-dimensional surface is created by the motion of a virtual particle dived in a stochastic harmonic potential. It brings out of the continuous RW that we can rigorously linked the partition function of the line to the Hamilton equation of motion for the transversal position. The survey of roughness amounts therefore to solve a KPZ equation with a particular potential. Then, interpreting the roughness as a result of a stochastic elastic forcing term acting on a surface, we have obtained a forced oscillator equation where the force behaves as a noise for the line. The analytical expressions of the pulsation and of the force have been determined too. Consequently, local analysis of the roughness may be performed thanks to the expression of the hvv (46) which, besides, obeys a Burgers equation. For a sufficient long substrate, the O-U model shows that the maximal values of roughness are asymptotically bounded in a range of length  $\Lambda \propto \tilde{\kappa}^{-1/2}$ . The elastic constant being an adjustable parameter mastering the magnitude of the fluctuations. Moreover, we have shown that the two models, used with a same kind of force, had common properties since they lead both to the same class of equations. As a matter of fact, the two equations of motion are connected to the partition function of the system. Finally, these two models belong to stochastic processes described by the Burgers dynamics.

Although we have presented one-dimensional models, these approaches could open interesting ways of investigation for numerical and analytical studies, in particular if one needs to generate surfaces with specific profiles or properties.

## APPENDIX: SOLUTION OF THE KPZ EQUATION

We solve here the KPZ equation (26) or, equivalently, the heat equation (25) written in the following way:

$$\frac{\partial}{\partial s} Z = \nu \frac{\partial^2 Z}{\partial y^2} + \left( -\frac{\beta\kappa}{2} y^2 + \beta\eta y \right) Z, \quad (\text{A.1})$$

with the initial condition  $Z(y, 0) = \exp(-u_0 y / 2\nu)$ . The resolution is based on the Time Space Transformation method (TST) of [51]. We give only the main points of resolution. We would like to underline for the comprehension that the notations  $x$  and  $y$  are reversed between the present paper and [51]:  $y \leftrightarrow x$ .

The first step is to determine the initial condition. For this, we need to determine the expression for the variables  $a_1$ ,  $a_2(s)$ ,  $p(s)$ ,  $q(s)$  and  $\tau(s)$  of [51]. We obtain successively:

$$a_1 = \sqrt{\kappa}/4\nu, \quad (\text{A.2})$$

$$a_2(s) = e^{-\sqrt{\kappa}s} \left( 1 + \beta \int_0^s \eta(t) e^{\sqrt{\kappa}t} dt \right), \quad (\text{A.3})$$

$$p(s) = e^{\sqrt{\kappa}s}, \quad (\text{A.4})$$

$$q(s) = 2\nu s + \int_0^s \int_0^t \eta(t') e^{\sqrt{\kappa}t'} dt' dt, \quad (\text{A.5})$$

$$\tau = e^{\sqrt{\kappa}s} \sinh(\sqrt{\kappa}s) / \sqrt{\kappa}. \quad (\text{A.6})$$

In such a way that the resolution of (A.1) amounts to solve the equation:

$$\frac{\partial \Phi}{\partial \tau} = \nu \frac{\partial^2 \Phi}{\partial x^2}, \quad (\text{A.7})$$

with the initial condition

$$\Phi(x, 0) = \exp(-a_1 x^2 - (1 + u_0/2\nu)x). \quad (\text{A.8})$$

The solution reads explicitly

$$\Phi(x, \tau) = (4a_1\nu\tau + 1)^{-1/2} \exp \left\{ -a_1 x^2 - (1 + u_0/2\nu)x + \frac{\nu\tau(2a_1x + 1 + u_0/2\nu)^2}{(4a_1\nu\tau + 1)} \right\}. \quad (\text{A.9})$$



Then, reversing the different transformations, we obtain as a solution of (A.1)

$$Z(y, s) = \exp \left\{ y^2 \left( a_1 - a_1 e^{2\sqrt{\kappa}s} + \frac{4a_1^2\nu}{\sqrt{\kappa}} e^{2\sqrt{\kappa}s} \tanh(\sqrt{\kappa}s) \right) + \right. \\ \left. y \left( a_2(s) - 2a_1q(s)e^{\sqrt{\kappa}s} - (1 + u_0/2\nu)e^{\sqrt{\kappa}s} + \frac{2\nu a_1}{\sqrt{\kappa}}(1 + u_0/2\nu + 2a_1q(s))e^{\sqrt{\kappa}s} \tanh(\sqrt{\kappa}s) \right) + \right. \\ \left. \left( -a_1q^2(s) + \frac{\nu}{\sqrt{\kappa}}(1 + 2a_1q(s))^2 \tanh(\sqrt{\kappa}s) - \frac{\sqrt{\kappa}}{2}s - \frac{1}{2} \ln(\cosh(\sqrt{\kappa}s)) \right) \right\} \quad (\text{A.10})$$

expression that we write for further analyse

$$Z(y, s) = \exp \left( \frac{\xi(s)}{4\nu} y^2 - \frac{\psi(s)}{2\nu} y + \frac{\Theta(s)}{2\nu} \right), \quad (\text{A.11})$$

where

$$\xi(s) = 4\nu \left( a_1 - a_1 e^{2\sqrt{\kappa}s} + \frac{4a_1^2\nu}{\sqrt{\kappa}} e^{2\sqrt{\kappa}s} \tanh(\sqrt{\kappa}s) \right), \quad (\text{A.12})$$

$$\psi(s) = -2\nu \left( a_2(s) - 2a_1q(s)e^{\sqrt{\kappa}s} - (1 + u_0/2\nu)e^{\sqrt{\kappa}s} + \frac{2\nu a_1}{\sqrt{\kappa}}(1 + u_0/2\nu + \right. \\ \left. + 2a_1q(s))e^{\sqrt{\kappa}s} \tanh(\sqrt{\kappa}s) \right), \quad (\text{A.13})$$

$$\Theta(s) = 2\nu \left( -a_1q^2(s) + \frac{\nu}{\sqrt{\kappa}}(1 + 2a_1q(s))^2 \tanh(\sqrt{\kappa}s) - \frac{\sqrt{\kappa}}{2}s - \frac{1}{2} \ln(\cosh(\sqrt{\kappa}s)) \right), \quad (\text{A.14})$$

The quantity  $a_1$ ,  $a_2(s)$  and  $q(s)$  being respectively given by Eqs. (A.2), (A.3) and (A.5). Note that we recover the initial condition  $Z(y, 0)$ . Then, since  $Z = \exp(\widehat{S}/2\nu)$ , we can write the solution on the final form

$$\widehat{S}(y, s) = \frac{\xi(s)}{2} y^2 - y\psi(s) + \Theta(s) \quad (\text{A.15})$$

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